Lecture 2: Fundamental Groupoid

Path connected components

Definition

Let $X \in$ **Top**.

- A map $\gamma : I \to X$ is called a path from $\gamma(0)$ to $\gamma(1)$.
- We denote γ^{-1} be the path from $\gamma(1)$ to $\gamma(0)$ defined by $\gamma^{-1}(t) = \gamma(1-t)$
- We denote $i_{x_0} : I \to X$ be the constant map to $x_0 \in X$.

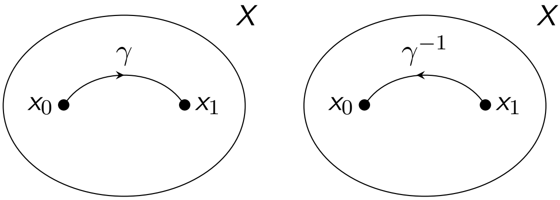


图: A path γ in a topological space X and its inverse

Let us introduce an equivalence relation on X by

$$x_0 \sim x_1 \iff \exists \text{ a path from } x_0 \text{ to } x_1.$$

We denote the quotient space

$$\pi_0(X) = X/\sim$$

which is the set of path connected components of X.

Theorem

$\pi_0 \colon \mathbf{hTop} \to \mathbf{Set}$ defines a covariant functor.

Corollary

If X, Y are homotopy equivalent, then $\pi_0(X) \simeq \pi_0(Y)$.

Path category / fundamental groupoid

Definition

Let $\gamma : I \to X$ be a path. We define the path class of γ by

$$[\gamma] = \{\tilde{\gamma} : I \to X | \gamma \simeq \tilde{\gamma} \ \mathrm{rel}\, \partial I = \{0, 1\}\}.$$

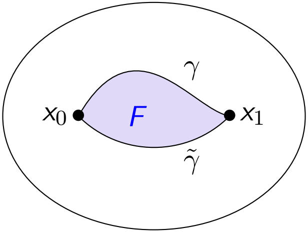


图: In a path class, F: $\gamma \simeq \tilde{\gamma} \ \mathrm{rel}\, \partial I$

$[\gamma]$ is the class of all paths that can be continuously deformed to γ while fixing the endpoints.

Definition

Let $\gamma_1, \gamma_2 : I \to X$ such that $\gamma_1(1) = \gamma_2(0)$. We define the composite path

$$\gamma_2 \star \gamma_1 : I \to X$$

by

$$\gamma_2 \star \gamma_1(t) = \begin{cases} \gamma_1(2t) & 0 \le t \le 1/2 \\ \gamma_2(2t-1) & 1/2 \le t \le 1, \end{cases}$$

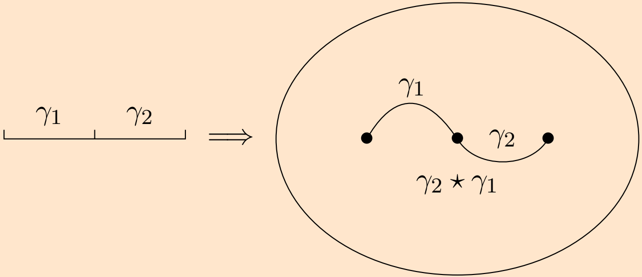


图: Composition of paths

Proposition

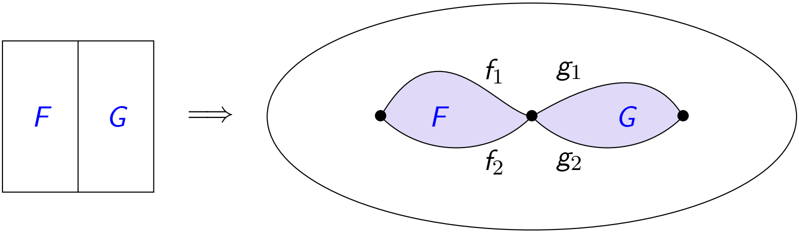
Let f_1, f_2, g_1, g_2 be paths, such that $f_i(1) = g_i(0)$, $[f_1] = [f_2]$, $[g_1] = [g_2]$. Then

$$[g_1 \star f_1] = [g_2 \star f_2].$$

Therefore $\star$ is well-defined for path classes.

Proof.

We illustrate the proof as follows



$\square$

Proposition (Associativity)

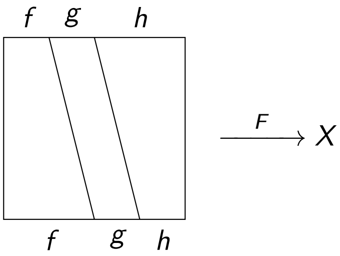
Let $f, g, h \colon I \to X$ with $f(1) = g(0)$ and $g(1) = h(0)$. Then

$$([h] \star [g]) \star [f] = [h] \star ([g] \star [f]).$$

Proof.

We illustrate the proof as follows

$[h] \star ([g] \star [f]) =$



$([h] \star [g]) \star [f] =$

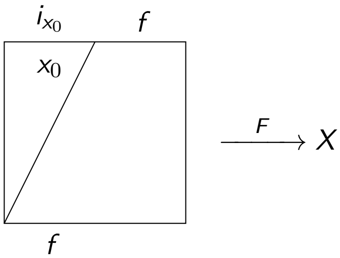
Proposition

Let $f\colon I \to X$ with endpoints $f(0) = x_0$ and $f(1) = x_1$. Then

$$[f] \star [i_{x_0}] = [f] = [i_{x_1}] \star [f].$$

Proof.

We only show the first equality, which follows from the figure



Definition

Let $X \in$ **Top**. We define a category $\Pi_1(X)$ as follows:

- $\mathrm{Obj}(\Pi_1(X)) = X$.
- $\mathrm{Hom}_{\Pi_1(X)}(x_0, x_1)$=path classes from x_0 to x_1.
- $1_{x_0} = i_{x_0}$.

The propositions above imply $\Pi_1(X)$ is a well-defined category. $\Pi_1(X)$ is called the path category or fundamental groupoid of X.

Groupoid

Definition

A category where all morphisms are isomorphisms is called a groupoid. All groupoids form a category $\underline{\mathbf{Groupoid}}$.

Example

A group G can be regard as a groupoid $\underline{G}$ with

- $\mathrm{Obj}(\underline{G}) = \{\star\}$ consists of a single object.
- $\mathrm{Hom}_{\underline{G}}(\star, \star) = G$ and composition is group multiplication.

Thus we have a fully faithful functor $\underline{\mathbf{Group}} \to \underline{\mathbf{Groupoid}}$.

Theorem

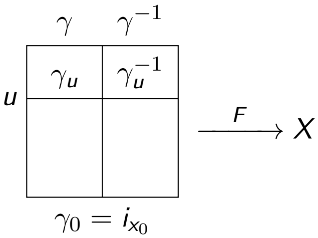
Let $\gamma\colon I \to X$ with endpoints $\gamma(0) = x_0$ and $\gamma(1) = x_1$. Then

$$[\gamma] \star [\gamma^{-1}] = [1_{x_1}], \quad \text{and} \quad [\gamma^{-1}] \star [\gamma] = [1_{x_0}].$$

Therefore $\Pi_1(X)$ is a groupoid.

Proof.

Let $\gamma_u\colon I \to X$ such that $\gamma_u(t) = \gamma(tu)$. The following figure gives the homotopy $\gamma^{-1} \star \gamma \simeq 1_{x_0}$:



Definition

Let $\mathcal{C}$ be a groupoid. Let $A \in \mathrm{Obj}(\mathcal{C})$, we define its automorphism group by

$$\mathrm{Aut}_{\mathcal{C}}(A) := \mathrm{Hom}_{\mathcal{C}}(A, A).$$

Note that this indeed forms a group.

$$\mathrm{Aut}_{\mathcal{C}}(A) = \left\{ \circlearrowleft A \right\}$$

For any $f\colon A \to B$, it induces a group isomorphism

$$\begin{aligned} \mathrm{Ad}_f\colon \mathrm{Aut}_{\mathcal{C}}(A) &\to \mathrm{Aut}_{\mathcal{C}}(B) \\ g &\to f \circ g \circ f^{-1}. \end{aligned}$$

Here is a figure to iillustrate

$$\mathrm{Ad}_f\colon \text{maps} \quad g \circlearrowright A \quad \text{to} \quad g \circlearrowright A \underset{f^{-1}}{\overset{f}{\rightleftarrows}} B$$

This naturally defines a functor

$$\mathcal{C} \to \underline{\mathbf{Group}} \quad \text{by assigning} \quad A \mapsto \mathrm{Aut}_{\mathcal{C}}(A), \quad f \mapsto \mathrm{Ad}_f.$$

Specialize this to topological spaces, we find a functor

$$\boxed{\Pi_1(X) \to \underline{\mathbf{Group}}}.$$

Definition

Let $x_0 \in X$, the group

$$\pi_1(X, x_0) := \mathrm{Aut}_{\Pi_1(X)}(x_0)$$

is called the fundamental group of the pointed space (X, x_0).

Theorem

Let X be path connected. Then for $x_0, x_1 \in X$, we have a group isomorphism

$$\pi_1(X, x_0) \simeq \pi_1(X, x_1).$$

Proof.

Consider the functor $\Pi_1(X) \to \textbf{Group}$ described above. Since X is path connected and $\Pi_1(X)$ is a groupoid, any two points x_0 and x_1 are isomorphic in $\Pi_1(X)$. Since functors preserves isomorphism, we conclude $\pi_1(X, x_0) \simeq \pi_1(X, x_1)$. □

In the path connected case, we will simply denote by $\pi_1(X)$ the fundamental group without mentioning the reference point.

Let $f: X \to Y$ be a continuous map. It defines a functor

$$\Pi_1(f) : \Pi_1(X) \to \Pi_1(Y) \quad \text{by assigning} \quad x \mapsto f(x), \quad [\gamma] \mapsto [f \circ \gamma].$$

Proposition

Π_1 defines a functor

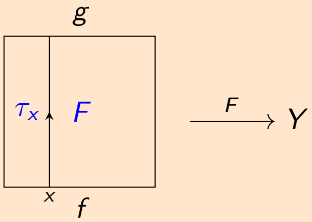
$$\boxed{\Pi_1 : \textbf{Top} \to \textbf{Groupoid}},$$

that sends X to $\Pi_1(X)$.

Proposition

Let $f, g : X \to Y$ be maps which are homotopic by $F : X \times I \to Y$. Let us define path classes

$$\tau_x = [F|_{x \times I}] \in \mathrm{Hom}_{\Pi_1(Y)}(f(x), g(x)),$$



Then τ defines a natural transformation

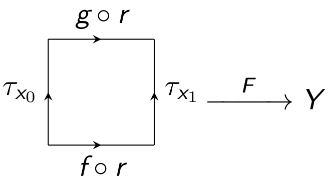
$$\tau_F \colon \Pi_1(f) \Longrightarrow \Pi_1(g).$$

Proof:

Let $r\colon I \to X$ with $r(t) = x_t$. We only need to show that the following diagram is commutative at the level of path classes:

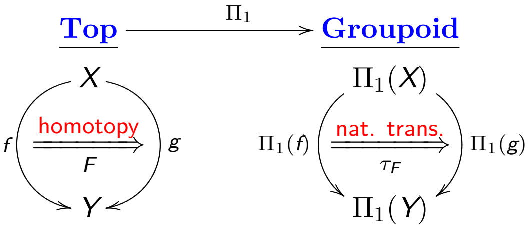
$$\begin{array}{ccccc}
f(x_0) = & \Pi_1(f)(x_0) & \xrightarrow{f\circ r} & \Pi_1(f)(x_1) & = f(x_1) \\
 & \downarrow \tau_{x_0} & & \downarrow \tau_{x_1} & \\
g(x_0) = & \Pi_1(g)(x_0) & \xrightarrow[g\circ r]{} & \Pi_1(g)(x_1) & = g(x_1)
\end{array}$$

The composition $F \circ (r \times I)$ gives the following diagram:



which implies that $[g \circ r] \star [\tau_{x_0}] = [\tau_{x_1}] \star [f \circ r]$ as required. $\square$

This proposition can be pictured by the following diagram



The following theorem is a formal consequence of the above proposition

Theorem

Let $f: X \to Y$ be a homotopy equivalence. Then

$$\Pi_1(f) : \Pi_1(X) \to \Pi_1(Y)$$

is an equivalence of categories. In particular, it induces a group isomorphism

$$\pi_1(X, x_0) \simeq \pi_1(Y, f(x_0)),$$

Proof.

Let $g\colon Y \to X$ represent the inverse of f in **hTop**. Applying Π_1 to $f \circ g \simeq 1_Y$ and $g \circ f \simeq 1_X$, we find $\Pi_1(f) \circ \Pi_1(g)$ and $\Pi_1(g) \circ \Pi_1(f)$ are natural isomorphic to identity functors. Thus the first statement follows.

The second statement follows from the fact that equivalence functors are fully faithful. □

Proposition

Let $X, Y \in \mathbf{Top}$. Then we have an isomorphism of categories

$$\Pi_1(X \times Y) \cong \Pi_1(X) \times \Pi_1(Y).$$

In particular, for any $x_0 \in X, y_0 \in Y$, we have a group isomorphism

$$\pi_1(X \times Y, x_0 \times y_0) \simeq \pi_1(X, x_0) \times \pi_1(Y, y_0).$$

Example

For a point $X = \text{pt}$, $\pi_1(\text{pt}) = 0$ is trivial. It is not hard to see that $\mathbb{R}^n$ is homotopy equivalent to a point. It follows that

$$\pi_1(\mathbb{R}^n) = 0 \quad n \geq 0.$$

Example

As we will see,

$$\pi_1(S^1) = \mathbb{Z}, \quad \text{and} \quad \pi_1(S^n) = 0, \forall n > 1.$$

Let $T^n = (S^1)^n$ be the n-dim torus. Then

$$\pi_1(T^n) = \mathbb{Z}^n.$$

Example (Braid groups)

Artin's braid group Br_n of n strings has the finite presentation:

$$\mathrm{Br}_n = \langle b_1, \ldots, b_{n-1} \mid \begin{array}{ll} b_i b_j b_i = b_j b_i b_j & \forall |j-i| = 1, \\ b_j b_i = b_i b_j & \forall\, |j-i| > 1 \end{array} \rangle.$$

Braid groups can be realized as fundamental groups.

The n^{th} (ordered) configuration space of X is

$$\mathrm{Conf}_n(X) := \{\underline{x} = (x_1, \ldots, x_n) \in X^n \mid x_i \neq x_j,\ \forall i \neq j\}.$$

It carries a natural action of the permutation group S_n

$$\begin{aligned} S_n \times \mathrm{Conf}_n(X) &\longrightarrow \mathrm{Conf}_n(X) \\ (\sigma, \underline{x}) &\longmapsto \sigma(\underline{x}) = (x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)}). \end{aligned}$$

The unordered configuration space of X is the orbit space :

$$\mathrm{conf}_n(X) = \mathrm{Conf}_n(X)/S_n.$$

A classical result says

$$\mathrm{Br}_n \cong \pi_1(\mathrm{conf}_n(\mathbb{R}^2)) \cong \pi_1(\mathrm{conf}_n(D^2)).$$

Fix n distinct points $Z_1, \cdots, Z_n$ in $\mathbb{R}^2$. A geometric braid is an n-tuple $\Psi = (\psi_1, \ldots, \psi_n)$ of paths

$$\psi_i\colon [0,1] \to \mathbb{R}^2$$

such that

- $\psi_i(0) = Z_i$;
- $\psi_i(1) = Z_{\nu(i)}$ for some permutation ν of $\{1, \ldots, n\}$;
- $\{\psi_1(t), \ldots, \psi_n(t)\}$ are distinct points in $\mathbb{R}^2$, for $0 < t < 1$.

The product of geometric braids follows the same way of products of paths (in the fundamental group setting). All braids on $\mathbb{R}^2$ with the product above form the braid group.

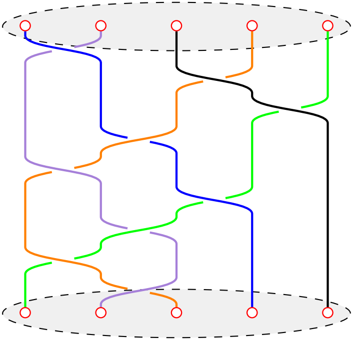


图: Classical braids